

Advanced Mathematics and Mathematical Engineering

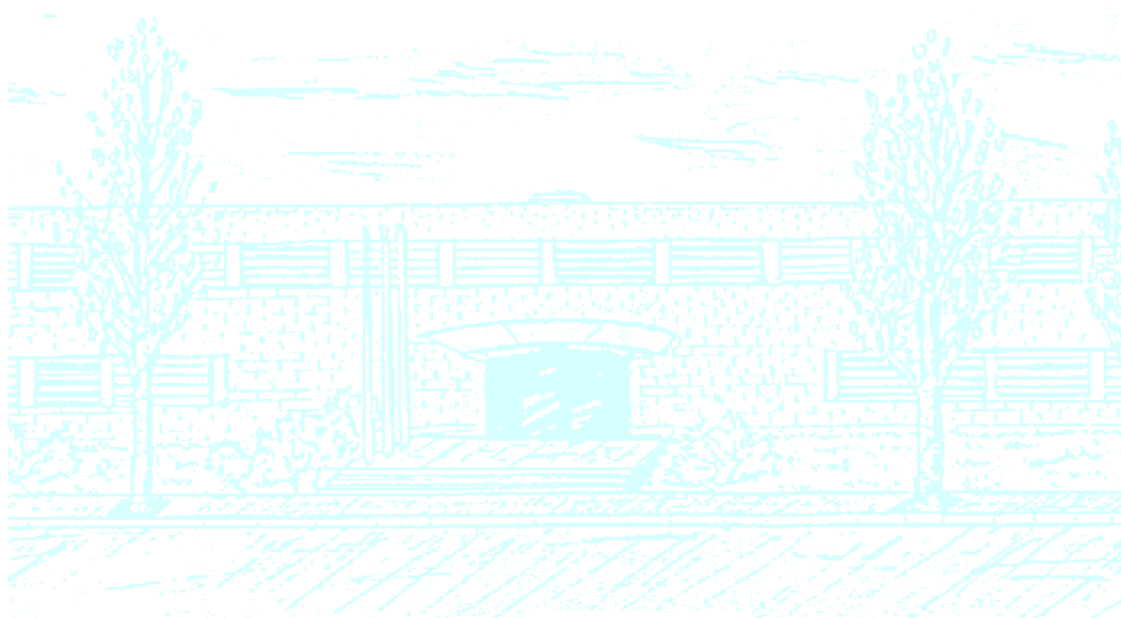
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Conic Portfolio Theory

Master's Degree Thesis

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1 Introduction

We are giving an insight to conic financial theory, a theory of financial analysis based on a two price economy for assets return, market valuation and risk measures. In particular, we centre our attention to portfolio optimisation with conic finance, where one must select the weights of each asset to maximise the value of a portfolio in the market. We follow, in principle, the exposition of conic portfolio theory from the recently published work by Madan [DM 15], and go in to deeper detail on computational issues and optimisation of portfolios under cardinality and other constraints.

In conic portfolio theory, risk and return are not two separated quantities as happens in classical theory of portfolio optimisation, but parameters that together generates the *bid price*, the objective function of the problem. In a broad sense, the bid price is the portfolio expected value for the worst future distribution of portfolio returns in a wide range of scenarios.

We define a distortion function $\Psi^\gamma : [0, 1] \longrightarrow [0, 1]$ as a concave function depending on a parameter γ called *stress level*. Such distortion gives rise to an alternative and more practical definition of bid price. The bid price can be seen as the expected value of portfolio future return with its current distribution distorted by Ψ^γ . One can use this formulation, apply a discretisation and get the portfolio that maximises the bid price.

The stress level is a parameter of the distortion function that measures the goodness or potential profitability of an asset. Hence, we say that Ψ^γ is a measure of risk. Indeed, it is a coherent risk measure which satisfy some intuitive properties that a risk measure must hold. For instance, it satisfies a quasi-concavity property, which says that a combination of two assets must have less risk than having just one asset or the other, therefore there is a *benefit of diversification*. This property together with the scale invariance of coherent risk measures let us define a *convex cone* of all possible return distributions, our scenarios.

Moreover, the stress level is very much related with the spread between bid and ask prices, which is an indicator of liquidity risk. A particular and differentiating feature of this level of risk is that it can be calibrated from market data, giving rise to a market risk level. In the classical sense, we try to maximise the return while minimising the risk of a portfolio, gauge by our personal tolerance, but in conic finance we consider the risk gauged by the market.

Finally, we get an implementable function to optimise. Let (R_1, \dots, R_N) be the returns of N assets. The problem we intend to solve consist on finding the weights $a_i \in [0, 1]$ of each asset that maximise the bid price of the portfolio, which depends on R_i . There

is a restriction on the weights for the problem to be consistent

$$\sum_{i=1}^N a_i = 1.$$

Further restrictions can be added to the problem depending on the investing preferences and on the type of portfolio. If you just want to hold long positions we add the constraint $a_i \geq 0$. Besides, one can limit the number of assets, the maximum and minimum weight and much more.

In the following sections we will bring light to some of the concepts mentioned above. In Chapter 2, we introduce the bid price function and the stress level as a measure of risk. In Chapter 3, we explain the optimisation problems and its computational complications. Chapter 4 presents a large analysis and simulations of conic portfolios. Here, we present a way of calibrating γ and compare conic portfolio selections under different distortion measures (MaxMinVar and Wang), and also against classical optimisation methods. Then, in Section 4.5 we calculate the weights under different constraints. Chapter 5 presents our conclusions.

2 Portfolio management

The process of investing consists on going to the stock market, choosing some assets and creating a portfolio trying to get the maximum profit. Finding a portfolio of maximum profit is not a trivial task at all and many theories have been developed. The main difficulty of the optimisation problem is valuating profitability beforehand as market oscillations could occur.

For example, in the so studied theory of Markowitz, profitability is measured with mean return. Nevertheless, more benefits usually imply assuming more risk of losses, hence the optimisation problem is a balance between risk and return selecting a risk level which the investor is willing to tolerate. Indeed, there are different risk estimators available and, in particular, volatility is used in Markowitz theory.

However, if a shareholder wants to recover his invested money, he will have to sell the portfolio and he wants the market to pay the maximum for it. Therefore, in conic portfolio theory an investor does not select his preferred assets, but the most valuable for the market so that he gets a better return. Therefore, the problem consists on finding the portfolio that maximises the price at which the market is willing to buy (conservative market value or bid price) with risk modelled by a stress level γ .

Looking at the market and not at the investor personal interests is quite useful. Not only for individual investors, but also for the problem facing a corporate entity like a pension fund or an insurance company. These companies manage portfolios on behalf of a large group of individuals with different risk attitudes, where a common risk level for all the investors can not be settled.

2.1 Study of the two price economy

Assets in a conic portfolio are not thought as having just one price, because in this situation it is not possible to model the benefit of diversification. This is a consequence of the fact that the valuation operators in an arbitrage free one price economy are linear [SR 78], where the price p of two assets X and Y must satisfy

$$p(X + Y) = p(X) + p(Y).$$

The linearity of these operators is deduced from the *non-arbitrage hypothesis*, meaning that there is no arbitrage¹ opportunities in the markets. If there was not equality in the previous equation, we could have bought two assets at a price and sell them more expensive, obtaining a return without risk.

We conclude that another valuation strategy have to be considered. A natural approach is to consider that there are two prices for every asset as happens in reality

¹There is a proper definition of arbitrage in the glossary.

in stock markets, where there is a price to sell to the market (bid price) and a price to buy from the market (ask price). In this section, we are going to develop this key idea used in conic portfolios, following the article by Madan [DM 15].

To work mathematically with assets we fix a horizon T at which we want to have benefits. Consider an initial and a final date $[t_0, T]$ and an asset A . We define Z as the cash flow of A between these two dates. At the initial date we do not know if the asset is going to report benefits, thus Z is a random variable on a probability space (Ω, \mathcal{F}, P) .

It is going to be useful to know the expectation of the price of an asset at the final date, which we call X , obtained assuming the non-arbitrage hypothesis:

$$E_q[X] = (1 + r)X^{(0)}, \quad (1)$$

where $X^{(0)}$ is the price at the initial date, r is the interest rate for the period and q is the *risk neutral* probability measure of X . This formula says that the expected value of an asset is the same as the profit obtained investing our money at interest rate r .

We can write the cash flows of A and use the expectation formula (1) to obtain the assets which are acceptable to invest in. The cash flow when purchasing an asset at price w is

$$Z = X - (1 + r)w, \quad (2)$$

with $w \leq X^{(0)}$, prices at which the market is willing to buy. Analogously, the cash flow when selling an asset at price w is

$$Z = (1 + r)w - X, \quad (3)$$

with $w \geq X^{(0)}$, prices at which the market is willing to sell A . For both cases the expectation of cash flows is positive:

$$E_q[Z] \geq 0.$$

So, we can define the assets acceptable to the market as the ones with cash flows in

$$\mathcal{A} = \{Z \mid E_q[Z] \geq 0\},$$

where Z depends on the probability q and on the current price of the asset. \mathcal{A} is a set of acceptable cash flows and, in particular, it is a convex cone.

A more general set of acceptable cash flows can be defined choosing a convex set of probability measures \mathcal{M} . In the literature, these measures $q \in \mathcal{M}$ are called test measures or generalised scenarios. The *acceptability cone*, in this case, is

$$\mathcal{A} = \{Z \mid E_q[Z] \geq 0, \quad \forall q \in \mathcal{M}\}. \quad (4)$$

Now, we have the tools to define the prices at which the market agrees to trade. The market agrees to buy an asset A at price b or sell it at price a if

$$X - b(1 + r) \in \mathcal{A}, \quad a(1 + r) - X \in \mathcal{A},$$

which means that for all $q \in \mathcal{M}$

$$\begin{aligned} E_q[X] - b(1 + r) &\geq 0, \\ a(1 + r) - E_q[X] &\geq 0. \end{aligned}$$

This leads to a bound for the bid b and ask a prices

$$\begin{aligned} b &\leq \frac{1}{1 + r} E_q[X_i], \\ a &\geq \frac{1}{1 + r} E_q[X_i]. \end{aligned}$$

The bid and ask prices are the best deals one can obtain from the market:

$$\begin{aligned} b(X) &= \frac{1}{1 + r} \inf_{q \in \mathcal{M}} E_q[X], \\ a(X) &= \frac{1}{1 + r} \sup_{q \in \mathcal{M}} E_q[X]. \end{aligned}$$

Remark: The bid and ask prices can change if more measures are considered in the set \mathcal{M} , even causing a change of the optimal portfolio. In particular, taking just one probability measure in the set \mathcal{M} makes the bid and ask prices equal, going back to one price economy.

An investor wants to maximise the bid price of their portfolio, to obtain greater benefit in case of selling. So, we consider as objective function the bid price without its multiplying constant

$$b(X) = \inf_{q \in \mathcal{M}} E_q[X], \tag{5}$$

which do not change the optimal result. The function $b(X)$ is a *coherent risk measure*, such measures will be studied in Section 2.2.

2.1.1 Bid price properties and distribution

The bid price defined in (5) is by construction concave as it is the infimum of a linear operator. This property is important because we can express the bid price of different assets together as

$$b(X + Y) = b(X) + b(Y) + D(X, Y), \tag{6}$$

where $D(X, Y) \geq 0$ is called the *benefit of diversification*. There is an advantage of selling assets together instead of separately. Nevertheless, if assets X, Y are comonotone

there is no such diversification benefit, as the losses of one asset are not compensated with the benefits of the other. In such case, the formula is rewritten as

$$b(X + Y) = b(X) + b(Y). \quad (7)$$

A function b satisfying formula (7) for every pair of comonotone random variables X and Y , is said to have the property of *comonotone additivity*.

Remark: One may think that formula (6) is giving rise to arbitrage opportunities, being possible to buy assets separately and sell them more expensive. But this is not the case, as one can only buy at ask price $a(X)$ and not at bid price $b(X)$. To be sure that there is no arbitrage opportunity we have to impose that

$$b(X) + b(Y) \leq b(X + Y) \leq a(X) + a(Y). \quad (8)$$

As it was proved by Kusuoka in [SK 01], under the assumption of comonotone additivity and assuming that the bid price depends only on the distribution of asset returns, then the bid price is an expectation applying a concave distortion. Being specific, there exists a concave function $\Psi(u)$ from the unit interval to itself such that for any random variable X with distribution $F_X(x)$ and density $f_X(x)$ its bid price is given by

$$b(X) = \int_{\mathbb{R}} x \, d\Psi(F_X(x)). \quad (9)$$

One can rewrite the expression and obtain

$$b(X) = \int_{\mathbb{R}} x \Psi'(F_X(x)) f_X(x) \, dx.$$

This shows that the bid price is an expectation of a new distribution. The change of density function and measure is given by

$$\Psi'(F_X(x)).$$

The choice of the concave function is of much interest as it determines the distribution of the bid price. Moreover, choosing Ψ is as important as selecting the set \mathcal{M} of risk measures²:

$$q \in \mathcal{M} \Leftrightarrow q(X) \leq \Psi(P(X)), \forall X \in \mathcal{F}.$$

Remark: It is not always true that the inherent probability measure induced by P belongs to \mathcal{M} . In other words, the distribution of asset returns $F_X(x)$ does not necessarily belong to the set of tested measures where the infimum is evaluated. This is inconvenient and can be solved by fixing the value of the concave function at $u = 0$ and $u = 1$ with

$$\Psi(0) = 0 \quad \text{and} \quad \Psi(1) = 1,$$

²See [MPS 12] for details.

which is a necessary condition for $\Psi(F_X(x))$ to be a probability distribution and is satisfied for the distortions Ψ that we are going to apply. There is also another distinguished probability contained in \mathcal{M} ; it was proved in [CGM01] that in absence of strictly acceptable opportunities a risk-neutral probability belongs to \mathcal{M} .

2.1.2 Data approximation of the bid price

Let (R_1, \dots, R_N) be a set of asset returns. The aim of conic theory is to find the portfolio that maximises the conservative market value. For this purpose, we define the portfolio return over an investment time horizon T as

$$R_p = \sum_{i=1}^N a_i R_i, \quad (10)$$

where a_i is the weight of the asset i and R_i its return, expressed as

$$R_i = \frac{X_i^{(f)} - X_i^{(0)}}{X_i^{(0)}},$$

with $X_i^{(0)}$ and $X_i^{(f)}$ the value of the asset i at initial and final dates. Formula (10) and the definition of the bid price (5) gives an expression for the bid price of a portfolio:

$$b(R_p) = \inf_{q \in \mathcal{M}} \left[\sum_{i=1}^N a_i \left(\frac{E_{q_i}[X_i^{(f)}]}{X_i^{(0)}} - 1 \right) \right], \quad (11)$$

where we have used that $X_i^{(0)}$ is known at initial time. For the optimisation problem defined with formula (11), we need to define the set \mathcal{M} that is considered. The most widely used methodology is to choose a particular type of parametrised distributions $q^\theta \in \mathcal{M}$ with parameters θ :

$$b(R_p) = \inf_{\theta_1, \dots, \theta_N} \left[\sum_{i=1}^N a_i \left(\frac{E_{q^{\theta_i}}[X_i^{(f)}]}{X_i^{(0)}} - 1 \right) \right]. \quad (12)$$

For example, one could take a lognormal distribution depending on $\theta = (\mu, \sigma)$ and look for the best parameters to fit the max-min problem, where the portfolio return will have a skewed lognormal distribution. However, this simplification does not really makes the problem easier to compute, although better than with a more general \mathcal{M} .

A different approach will be convenient: we are going to discretise formula (9) obtained by Kusuoka. Using that a sample applied to its distribution function is the uniform distribution one finds the expression

$$b(R_p) = \sum_{m=1}^M R_{p,(m)} \left(\Psi \left(\frac{m}{M} \right) - \Psi \left(\frac{m-1}{M} \right) \right), \quad (13)$$

with $R_{p,(m)}$ the sample returns of the portfolio arranged in increasing order. Substituting the portfolio return by formula (10) we obtain

$$b(R_p) = \sum_{m=1}^M \left(\sum_{i=1}^N a_i R_{i,(m)} \right) \left(\Psi \left(\frac{m}{M} \right) - \Psi \left(\frac{m-1}{M} \right) \right), \quad (14)$$

with $R_{i,(m)}$ the returns of every asset, where either historical data or a distribution simulation can be used to generate the sample. In the first case, we use samples of returns that have occurred in the past. In the second case, we assume that the returns of the assets are behaving in a particular way, i.e. following a distribution.

Remark: As we have already seen, the bid price equation (13) is equivalent to the original definition of bid. The advantage of this approach in comparison with (11) is that we do not choose a particular parametrised distribution, which is very restrictive, but we take distortions on the actual distribution function. For that we consider better to use historical data than a parametrised distribution, so that we do not assume any type of distribution for assets returns.

One can express the equations of the bid price with matrices using the formula

$$b(R_p) = \sum_{m=1}^M \left(\sum_{i=1}^N a_i R_{i,(m)} \right) \underbrace{\left(\Psi^\gamma \left(\frac{m}{M} \right) - \Psi^\gamma \left(\frac{m-1}{M} \right) \right)}_{c_m},$$

and rearranging it as follows

$$b(R_p) = \begin{pmatrix} c_1 & \dots & c_M \end{pmatrix} \cdot \begin{pmatrix} R_{1,1} & \dots & R_{1,N} \\ | & & | \\ R_{M,1} & \dots & R_{M,N} \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ | \\ a_N \end{pmatrix}. \quad (15)$$

The rows of the matrix $(R_{m,i})_{m,i}$ are ordered to make the function

$$\sum_{i=1}^N a_i R_{i,m}$$

increasing in m . So, we can write a formula for the bid price as follows

$$\begin{aligned} b(R_p) &= \begin{pmatrix} c_1 & \dots & c_M \end{pmatrix} \cdot \text{order} \left(\begin{pmatrix} R_{1,1} & \dots & R_{1,N} \\ | & & | \\ R_{M,1} & \dots & R_{M,N} \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ | \\ a_N \end{pmatrix} \right) \\ &= \begin{pmatrix} c_1 & \dots & c_M \end{pmatrix} \cdot \text{order} \left(\begin{pmatrix} \sum_{i=1}^N a_i R_{1,i} & \dots & \sum_{i=1}^N a_i R_{M,i} \end{pmatrix} \right). \end{aligned} \quad (16)$$

These are the formulas that are going to be used in the simulations.

2.2 Portfolio measures of performance

The first step when investing is deciding which assets to invest in. Selecting the bigger and more stable enterprises, for example companies that are part of a market index, for instance Ibex 35 index in the Spanish market, is not always a guarantee.

We would like to choose the best companies of the market. To proceed in this way, one should have an indicator of the quality of an enterprise, we call such indicators *measures of performance*. A measure of performance is a function

$$\begin{aligned} \alpha : L^\infty(\Omega) &\longrightarrow [0, \infty] \\ Z &\longmapsto \alpha(Z) \end{aligned} \quad (17)$$

being Z a random variable representing the cash flow of the measured asset and Ω the sample space.

2.2.1 Coherent risk measures and indexes of acceptability

Coherent risk measures are of much interest for the conic portfolio study. In the theory of measures, a *coherent risk measure* is a function of the form

$$\rho(Z) = - \inf_{q \in \mathcal{M}} E_q[Z]. \quad (18)$$

As conic financial theory deals with bid prices, we are going to develop measure theory with the bid price $b(Z)$ instead of using a general coherent risk measure $\rho(Z)$.

A *pricing kernel* is a distribution $q \in \mathcal{M}$ such that $E_q[Z] = 0$, for every asset with no trade or no liquidity. We define the *set of supporting kernels* as

$$\mathcal{M} = \{q \in \mathcal{P} \mid E_q[Z] \geq b(Z), \quad \forall q \in L^\infty(\Omega)\},$$

where \mathcal{P} is the set of all possible probability measures. It is also interesting to know the *set of extreme measures* $Q^*(Z)$ defined as all the probability distributions q that attain the minimum, i.e.

$$E_q[Z] = b(Z), \quad \forall q \in Q^*(Z).$$

If $Q^*(Z)$ is a singleton, we call its only measure q^* and the set of supporting kernels can be obtained by doing convex combinations of $q^*(Z)$.

We say that a trade is acceptable when $b(Z) \geq 0$, being Z the cashflows of the trade. We define an acceptability set as

$$\mathcal{A} = \{Z \in L^\infty(\Omega) \mid b(Z) \geq 0\}.$$

We state some properties that will relate the bid price with risk levels, following [CM07].

1. Quasi-concavity

For a performance measure α , it is natural to define the set of acceptable trades at level x as

$$\mathcal{A}_x = \{Z \mid \alpha(Z) \geq x\}, \quad x \in \mathbb{R}^+$$

Quasi-concavity means that all these sets must be convex. Thus, the quasi-concavity property says that if two assets are acceptable at level x and you take a proportion of each, then the portfolio obtained is also acceptable at level x :

$$\alpha(X) \geq x \text{ and } \alpha(Y) \geq x, \text{ then } \alpha(\lambda X + (1 - \lambda)Y) \geq x \quad \forall \lambda \in [0, 1].$$

This property together with scale invariance makes the set \mathcal{A}_x to be a convex cone.

2. Monotonicity

It seems mandatory for a well defined measure of performance, to ask that if X is acceptable at a level and Y dominates X , then Y is acceptable at the same level. The monotonicity property requires this to be satisfied at every level:

$$\text{if } X \leq Y, \text{ then } \alpha(X) \leq \alpha(Y).$$

3. Scale invariance

When evaluating trades there are two main focus of attention: the direction of the trade and its size or scale. By the scale invariance property we say that a trade is acceptable by looking at the direction of the market and not the volume negotiated. Some people would argue that this is not consistent because more volume means assuming more risk, but we are just focusing on how good an asset is.

So, scale invariance requires that the measure $\alpha(Z)$ does not change under scaling

$$\alpha(\lambda Z) = \alpha(Z) \quad \text{for } \lambda > 0.$$

4. Fatou property

To deduce some interesting properties on a measure of performance a continuity property (Fatou property) is needed. It requires that if (Z_n) is a sequence of random variables such that $|Z_n| \leq 1$, $\alpha(Z_n) \geq x$ and Z_n converges to Z in probability, then $\alpha(Z) \geq x$.

A measure of performance with these four properties is called an *index of acceptability*. There are four extra properties that are important in conic portfolio theory and we want them to be satisfied by our index of acceptability.

5. Law invariance

If two cash flows have the same probability distribution, they should have the

same level of acceptability. This property means that the distribution of cash flows is the only factor that matters when setting the level of acceptability.

$$\text{if } X \stackrel{\text{law}}{=} Y, \text{ then } \alpha(X) = \alpha(Y),$$

where $X \stackrel{\text{law}}{=} Y$ means that X and Y have the same probability distribution.

6. Consistency with second order stochastic dominance

It is reasonable to require that if one trade is preferred to another by every market participant, then it should have a higher level of acceptability. If the participants' preferences are described by an expected utility U , then we get the property

$$\text{if } Y \text{ second-order stochastically dominates } X, \text{ then } \alpha(X) \leq \alpha(Y),$$

where second-order stochastic dominance means that $E[U(X)] < E[U(Y)]$ for any concave function U .

7. Arbitrage consistency

An arbitrage is a trading opportunity with zero risk, which means that its cash flow is a positive random variable. Arbitrages are universally acceptable and its level of acceptability should be set at infinity:

$$Z \geq 0 \text{ if and only if } \alpha(Z) = \infty.$$

8. Expectation consistency

This property says that if we expect losses from our asset, then the level of acceptability must be zero. It deals with the low values of the performance measure and requires that

$$\alpha(Z) = \begin{cases} 0 & \text{if } E[Z] \leq 0, \\ \alpha > 0 & \text{if } E[Z] > 0. \end{cases}$$

Let $(\mathcal{M}_x)_{x \in \mathbb{R}_+}$ be a family of supporting kernels satisfying $\mathcal{M}_x \subset \mathcal{M}_y$ for $x \leq y$. From this family of kernels we can define a coherent risk measure

$$b_x(Z) = \inf_{q \in \mathcal{M}_x} E_q[Z],$$

so $(b_x(Z))_{x \in \mathbb{R}_+}$ is a family of decreasing bid prices. Then, an acceptability index can be defined as

$$\alpha(Z) = \sup\{x \mid b_x(Z) \geq 0\}. \quad (19)$$

Analogously from coherent risk measures, we can define a *system of supporting kernels* as

$$\mathcal{M}_x = \{q \in \mathcal{P} \mid E_q[Z] \geq 0, \quad \forall Z \text{ such that } \alpha(Z) > x\},$$

and the *extreme system* as $(Q_x^*(Z))_{x \in \mathbb{R}_+}$ where every set contains the probability measures $q \in \mathcal{M}_x$ that attains the minimum. If there is just one distribution in $Q_x^*(Z)$ we call it q_x^* and

$$b_x(Z) = E_{q_x^*}[Z].$$

It have been shown in [CM 07] that: given $q^*(X) = q_{\alpha(X)}^*(X)$, for all $Y \in L^\infty$ we have

$$\begin{aligned} E_{q^*}(Y) > 0 &\Rightarrow \exists \delta > 0 \text{ such that } \forall \varepsilon \in (0, \delta) \quad \alpha(X + \varepsilon Y) > \alpha(X), \\ E_{q^*}(Y) < 0 &\Rightarrow \exists \delta > 0 \text{ such that } \forall \varepsilon \in (0, \delta) \quad \alpha(X + \varepsilon Y) < \alpha(X). \end{aligned}$$

This is saying that a trade Y acceptable at a level $\alpha(X)$ improve the current acceptability level. Hence, it seems reasonable to use $q^*(X)$ as the price distribution for X .

To solve the portfolio optimisation problem, a concrete acceptability index is needed. We are going to explain some well known indexes that will be tested in this project.

WVaR acceptability indexes

We define the TVaR coherent risk measure as

$$\text{TVaR}_\lambda(X) = - \inf_{q \in \mathcal{M}_\lambda} E_q[X]. \quad (20)$$

It was shown in [FS 04] that if X is a continuous distribution, then we can express the TVaR measure as

$$\text{TVaR}_\lambda(X) = E_q[X | X \leq x_\lambda(X)], \quad (21)$$

being $x_\lambda(X)$ the λ -quantile of X . In this alternative definition the TVaR measure is interpret as the losses of the assets in the worst cases. We call such measure *Tail VaR*, *Conditional VaR* or *Expected Shortfall*.

A generalisation of TVAR is the weighted value at risk WVAR, obtained from a one parameter family of TVaR measures. It is defined as the average of TVaR_λ with different risk levels λ , weighted by a probability measure μ

$$\text{WVAR}_\mu(X) = \int_0^1 \text{TVaR}_\lambda(X) \mu(d\lambda), \quad (22)$$

which is also a coherent risk measure. By the introduction of a concave function of the form

$$\Psi^\mu(y) = \int_0^y \int_z^1 \lambda^{-1} \mu(d\lambda) dz.$$

it was proved in [FS04] that the WVAR can be expressed as

$$\text{WVAR}_\mu(X) = - \int_{\mathbb{R}} y \, d(\Psi^\mu(F_X(y))). \quad (23)$$

In particular, a one parameter family Ψ^γ is chosen. This formulation let us define an index of acceptability for the WVAR measure as

$$\text{AIW}(X) = \sup\{\gamma \mid b_\gamma(X) = \int_{\mathbb{R}} y \, d(\Psi^\gamma(F_X(y))) \geq 0\}.$$

The $\text{AIW}(X)$ acceptability index is the maximum stress level at which X is acceptable. Hence, the risk of a portfolio can be measured with this stress level γ . The greater the stress level, the greater the index of acceptability and the market accepts less risk.

2.2.2 Stress level and calibration

We ought to measure the level of acceptability of the market to adjust the risk of our portfolio to market risk. We will deal with formula (9), where the bid price is obtained by a concave distortion of a probability measure. Parametric distortions are widely used in this case, with a parameter γ which is called *stress level*.

MAXMINVAR

In the article of Cherny and Madan [CM07] they conclude that MinMaxVar and MaxMinVar provide similar results and above of the ones provided by MaxVar and MinVar. In our simulations, we are going to use the concave function

$$\Psi^\gamma(u) = 1 - \left(1 - u^{\frac{1}{1+\gamma}}\right)^{1+\gamma}, \quad (24)$$

which leads to the *MaxMinVar* acceptability index. To construct an acceptability index we need that the distortion Ψ^γ increases with γ , which is satisfied by our Ψ^γ :

$$\begin{aligned} \gamma_1 < \gamma_2 &\Rightarrow \frac{1}{1+\gamma_1} > \frac{1}{1+\gamma_2} \Rightarrow u^{\frac{1}{1+\gamma_1}} < u^{\frac{1}{1+\gamma_2}} \text{ for } u \in [0, 1] \Rightarrow \\ 1 - u^{\frac{1}{1+\gamma_1}} &> 1 - u^{\frac{1}{1+\gamma_2}} \Rightarrow \left(1 - u^{\frac{1}{1+\gamma_1}}\right)^{1+\gamma_1} > \left(1 - u^{\frac{1}{1+\gamma_2}}\right)^{1+\gamma_2} \Rightarrow \\ 1 - \left(1 - u^{\frac{1}{1+\gamma_1}}\right)^{1+\gamma_1} &< 1 - \left(1 - u^{\frac{1}{1+\gamma_2}}\right)^{1+\gamma_2} \end{aligned}$$

This index depends on one parameter that should be calibrated from the market, this is done in Section 2.2.2.

Wang transform

We also introduce the Wang transform for its applications in financial theory. In particular, when pricing options using conic theory it is quite useful. In the framework of portfolio optimisation it have not been studied, so we will give some light into it.

The Wang transform is a concave distortion of the form

$$\Psi_{\Phi}^{\gamma}(u) = \Phi(\Phi^{-1}(u) + \gamma). \quad (25)$$

where Φ is the standard normal distribution function. As always, we have to check that the distortion increases with γ . This is trivial as any distribution, and in particular Φ , is an increasing function. Hence,

$$\gamma_1 < \gamma_2 \Rightarrow \Phi(\Phi^{-1}(u) + \gamma_1) < \Phi(\Phi^{-1}(u) + \gamma_2)$$

As stated in [KNS09] by applying this distortion one gets an index of acceptability with the feature that a distortion of a normal or log-normal distribution is again of the same type. Indeed, it is the only distortion satisfying these properties.

Calibrating market stress level

Given a market with N stocks and a horizon date T , we calibrate its stress level by an adjustment from real data, using historical data of the N stocks. In particular, end of day data of the previous T days. The algorithm is the following:

- 1.- For each of the N stocks:
 - (i) Estimate the bid price (b') and the ask price (a') from market data.
 - bid price (b'): the minimum price of the previous T days.
 - ask price (a'): the maximum price of the previous T days.
 - (ii) Relativize each of the previous estimated quantities to the average price (\bar{x}) of the previous T days:

$$b = \frac{b'}{\bar{x}}, \quad a = \frac{a'}{\bar{x}}.$$

This step settle the mid-quote price at 1 to allow comparison of the bid discount (b) and ask add on (a) among stocks.

- 2.- For each calibration date t : take the average of the bid b and ask a along the stocks. This average can be done considering the negotiated volume of each asset

$$b_t = \sum_{i=1}^N a_i b_t^{(i)},$$

where $b_t^{(i)}$ is the bid price of the asset i at calibration date t and a_i is the percentage negotiated volume of asset i . This procedure let the calibration of γ fit the current risk of the market.

- 3.- The last step is finding the stress level that best approximates the market bid price obtained in step 2. We estimate the stress level with least squares minimisation:

$$\gamma_t = \arg \min_{\gamma} \left(b_t - \hat{b}_t(\gamma) \right)^2 + (a_t - \hat{a}_t(\gamma))^2, \quad (26)$$

where

$$\begin{aligned} \hat{b}_t(\gamma) &= b_t(X, \gamma) = \sum_{m=1}^M x_{(m)} \left(\Psi^{\gamma} \left(\frac{m}{M} \right) - \Psi^{\gamma} \left(\frac{m-1}{M} \right) \right), \\ \hat{a}_t(\gamma) &= a_t(X, \gamma) = \sum_{m=1}^M x_{(m)} \left(\Psi^{\gamma} \left(\frac{m}{M} \right) - \Psi^{\gamma} \left(\frac{m-1}{M} \right) \right), \end{aligned}$$

with $x_{(m)} = \sum_{i=1}^N a_i R_i$ the average of asset returns at each date t , in increasing order for the bid price and in decreasing order for the ask price.

2.3 Risk on conic portfolio theory: sensibility of the stress level

A drawback one finds when applying this theory is that the market risk γ that we are selecting during the calibration may change due to market conditions. In particular, in financial crisis the stress level increases quickly with respect to past levels. This change on γ might make our portfolio non optimal in the new levels of stress of the market.

To have an idea of how much affectation will we have on changes of stress level we suggest to compute

$$V = \frac{\partial b(R_p, \gamma)}{\partial \gamma}, \quad (27)$$

which is an analogous for the Vega of option hedging.

To compute such derivative with real data one have to do a discretisation, as explicit expressions are not treatable. So, the expression for the sensibility on γ only depend on the partial derivative of the distortion function $\frac{\partial \Psi^{\gamma}}{\partial \gamma}$ in the following way

$$\frac{\partial b(R_p, \gamma)}{\partial \gamma} = \sum_{m=1}^M R_{p,(m)} \left(\frac{\partial \Psi^{\gamma}}{\partial \gamma} \left(\frac{m}{M} \right) - \frac{\partial \Psi^{\gamma}}{\partial \gamma} \left(\frac{m-1}{M} \right) \right), \quad (28)$$

with $R_{p,(m)}$ the sample returns of the portfolio arranged in increasing order.

Therefore, knowing the formula of the partial derivative of the distortion with respect

to γ is enough to obtain this measure. For instance, one can compute the value of such derivative for the MaxMinVar distortion

$$\frac{\partial \Psi^\gamma}{\partial \gamma}(x) = - \left(1 - x^{\frac{1}{1+\gamma}}\right)^{1+\gamma} \cdot \left(\frac{x^{\frac{1}{1+\gamma}} \cdot \log(x)}{(1+\gamma) \left(1 - x^{\frac{1}{1+\gamma}}\right)} + \log \left(1 - x^{\frac{1}{1+\gamma}}\right) \right). \quad (29)$$

If, on the other hand, we want to compute the sensibility on the Wang transform stress level the formula is

$$\frac{\partial \Psi^\gamma}{\partial \gamma}(x) = \phi \left(\Phi^{-1}(x) + \gamma \right), \quad (30)$$

where Φ is the standard normal distribution function and ϕ its density.

3 Optimisation problem

We want to find a portfolio that maximises its conservative market value or bid price. Indeed, the optimisation problem to be solved consists on finding the weights a_i of each asset to maximise the bid price. For a portfolio R_p defined as in (10) the optimisation problem is stated as follows

$$\begin{aligned} \text{find: } & \max_{a_i} b(R_p), \\ \text{subject to: } & \sum_{i=1}^N a_i = 1. \end{aligned} \quad (31)$$

We use the discretised formula (14) found in Section 2.1.2 to be able to compute the optimal solution. The main difficulty when solving the problem is that the objective function is non-linear due to the reordering of coefficients c_m in formula (16).

We must choose if we want to calibrate the parameters γ before doing the optimisation, as done in Section 2.2.2, or if we want to solve the optimisation problem with γ as parameters to be optimised. We will calibrate γ beforehand because it have more sense to consider a stress level for the whole market and not particularly for every portfolio. So, our problem is

$$\begin{aligned} \text{find: } & \max_{a_i} (c_1 \dots c_M) \cdot \text{order} \left(\sum_{i=1}^N a_i R_{1,i} \dots \sum_{i=1}^N a_i R_{M,i} \right), \\ \text{subject to: } & \sum_{i=1}^N a_i = 1, \end{aligned} \quad (32)$$

with

$$c_m = \Psi^\gamma \left(\frac{m}{M} \right) - \Psi^\gamma \left(\frac{m-1}{M} \right), \quad (33)$$

and γ calibrated from market data.

3.1 Long-only portfolio

We consider a portfolio restricted so that there is only the possibility to buy assets. Hence, we have to add the constrain $a_i \geq 0$ in our optimisation problem:

$$\begin{aligned} \text{find: } & \max_{a_i} b(R_p(\vec{a})), \\ \text{subject to: } & \sum_{i=1}^N a_i = 1, \\ & a_i \geq 0 \quad \forall i \in \{1, \dots, N\}. \end{aligned} \quad (34)$$

To solve this problem one can fix the average return and create an Efficient Frontier as will be done in the next section. The idea applied by Madan to solve this problem

3 OPTIMISATION PROBLEM

in [DM 15] is looking at how the bid price differs from the average return. We express the asset return over an investment time horizon T as

$$R_i = \mu_i + R_i^\varepsilon, \quad (35)$$

with μ_i the expected return and R_i^ε the centred return. Then, using properties of the bid function b , we find that the bid price of an asset is

$$b(R_i) = \mu_i + b(R_i^\varepsilon). \quad (36)$$

Analogously, we can also define the return of a portfolio as

$$R_p = \sum_{i=1}^N a_i R_i = \sum_{i=1}^N a_i (\mu_i + R_i^\varepsilon) = \sum_{i=1}^N a_i \mu_i + \sum_{i=1}^N a_i R_i^\varepsilon =: \mu_p + R_p^\varepsilon$$

and the bid price of a portfolio as

$$b(R_p) = \vec{a} \cdot \vec{\mu} + b(R_p^\varepsilon), \quad (37)$$

with $\vec{a} = (a_1, \dots, a_N)$ the vector of weights, $\vec{\mu} = (\mu_1, \dots, \mu_N)$ the vector of average returns and $R_p^\varepsilon = \sum_{i=1}^N a_i R_i^\varepsilon$ the centred returns. We would like the assets of our portfolio to be acceptable by their own. We know that an asset is acceptable if and only if its bid price is positive:

$$b(R_i) = \mu_i + b(R_i^\varepsilon) \geq 0,$$

so, we obtain that

$$\mu_i \geq -b(R_i^\varepsilon).$$

Then, the minimum average return for the asset to be acceptable is

$$\mu_i = -b(R_i^\varepsilon). \quad (38)$$

The problem finally is stated as follows

$$\begin{aligned} \text{find: } & \max_{a_i} \vec{a} \cdot \vec{\mu} + b(R_p^\varepsilon), \\ \text{subject to: } & \sum_{i=1}^N a_i = 1, \\ & a_i \geq 0 \quad \forall i \in \{1, \dots, N\}. \end{aligned} \quad (39)$$

We have a non-linear optimisation problem with a linear term depending on the average return and a non-linear part due to the reordering of coefficients. The point of using centred returns is that it makes easier the comparison of assets with different averages, which is useful from the financial analysis perspective.

3.2 Long-short portfolio

In a long-short portfolio there are no restrictions on the sign of weights, and hence, we have an optimisation problem under an unbounded domain which can lead to a non definite solution, with infinite weights. A further study is needed to make the optimisation problem treatable and finite.

To study the long-short portfolio we will look at how the bid price differs from the average return as done in the long-only portfolio. We express the asset return over an investment time horizon T as in (35) and we apply the same procedure to obtain the bid price formula (36).

The bid price for centred returns is a function depending on the weights of the assets, so we call it

$$c(\vec{a}) = b(R_p^\varepsilon). \quad (40)$$

Assuming that the base measure is in \mathcal{M} , it follows that $c(a)$ is a negative function as

$$c(a) = b(R_p^\varepsilon) = b(R_p) - \mu_p = \inf_{q \in \mathcal{M}} E_q[R_p] - E_q[R_p] \leq 0,$$

where $\mu_p = \vec{a} \cdot \vec{\mu}$ is the average return of the portfolio. In particular, $c(\cdot)$ can be turned into a positive function

$$\tilde{c}(\vec{a}) = -c(\vec{a}), \quad (41)$$

which is interpreted as the ask price for the contrary portfolio return, meaning that what was sold in our portfolio R_p is now bought and vice versa:

$$\tilde{c}(\vec{a}) = - \inf_{q \in \mathcal{M}} E_q[R_p^\varepsilon] = \sup_{q \in \mathcal{M}} E_q[-R_p^\varepsilon]. \quad (42)$$

In the end, the bid price can be expressed as

$$b(R_p) = \vec{a} \cdot \vec{\mu} - \tilde{c}(\vec{a}). \quad (43)$$

For the numerical computation of the portfolio, we fix an expected return that we want to achieve μ_p with our portfolio. Therefore, the optimisation problem is

$$\begin{aligned} \text{find: } & \min_{a_i} \tilde{c}(\vec{a}), \\ \text{subject to: } & \vec{a} \cdot \mathbf{1} = 1, \\ & \vec{a} \cdot \vec{\mu} = \mu_p. \end{aligned} \quad (44)$$

Solving this optimisation problem for different values of μ_p one obtains an ask price mean return Efficient Frontier. Recall that the weights obtained from this problem have the contrary sign of the ones of the original optimisation problem.

Once we have the Efficient Frontier, we would like to determine the value that maximises the portfolio, without the restriction of fixed return μ_p . For this, we draw a

unity slope line and look for the maximising of these lines on the Efficient Frontier, a tangency line analogous to the Capital Market line in Sharpe theory. The unity line has the following expression

$$\mu_p = c(\vec{a}) + b(R_p).$$

with the bid price as its independent term. Therefore, the maximising bid price is given by the maximal unity line. We can see an illustrative picture of this maximisation on the Efficient Frontier in Figure 1.

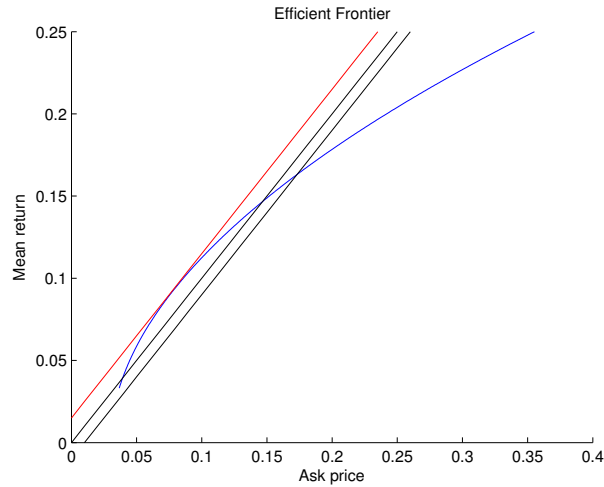


Figure 1: Mean Return Ask Price Efficient Frontier. The optimisation of the bid price is done increasing the unity lines. The red line is giving the optimal at the crossing with the Efficient Frontier (in blue).

3.3 Restricted optimisation problem

Sometimes an investor wants a particular type of portfolio. To deal with this situation one can add extra restrictions. These restrictions are applied to the problem due to investment decisions and can change the optimal portfolio. For example, one investor might want to have a maximum of assets to keep control of them, have a maximum of operations due to law or investment policy of his company or have a limitation in the weight of every asset, as diversification is considered a good practice.

- (a). **Bounds on holdings:** it is a usual practise to limit the weight of every asset a_i to be sure that the portfolio is distributed on a wide range of assets. This restriction is expressed as

$$L_i \leq a_i \leq U_i,$$

with L_i the lower bound and U_i the upper bound.

- (b). **Maximum trading:** these constraints impose upper bounds in the variation of assets that one can hold between periods of investing. They are modelled as follows

$$\begin{aligned} \max[a_i - a_i^{(0)}, 0] &\leq B_i \quad (\text{purchase}) \\ \max[a_i^{(0)} - a_i, 0] &\leq S_i \quad (\text{sale}) \end{aligned}$$

with $a_i^{(0)}$ the initial weight, B_i the maximum possible purchase and S_i the maximum sale.

- (c). **Minimum trading:** these constraints impose lower bounds of the variation of assets in the portfolio between investing times. They are modelled as follows

$$\begin{aligned} a_i - a_i^{(0)} &\geq B_i \quad (\text{purchase}) \\ a_i^{(0)} - a_i &\geq S_i \quad (\text{sale}) \end{aligned}$$

with $a_i^{(0)}$ the initial weight, B_i the minimum obligatory purchase and S_i the minimum obligatory sale.

- (d). **Size of the portfolio:** we can impose a limitation in the number of assets, as one may not want to keep track of every asset in the market, but just of K assets. Given $K \leq N$, we ask that

$$|\{i \in \{1, \dots, N\} \mid a_i \neq 0\}| \leq K.$$

To impose this restriction to the problem we define the integer variables

$$s_i = \begin{cases} 1 & a_i \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

and add the constraint

$$\sum_{i=1}^N s_i \leq K.$$

- (e). **Obligatory deposit constraint:** it says that an investor must give money as a deposit if he wants to hold a position, in particular, if he wants to hold a short position. This is done to assure that the investor will be able to buy the short asset if it increases in value. A usual case is to set the deposit $\delta_i = 1$ for long positions (you need all the money to buy the asset) and $\delta_i = \delta \in [0, 1]$ for every short position. The restriction is the following

$$\sum_{i=1}^N \delta_i |a_i| = 1,$$

with δ_i the proportion of deposit of the asset i . This is a non linear restriction that makes the problem fairly more difficult. As stated in [GK 99], with

this restriction the optimal solution is very sensitive to small changes in the parameters δ_i . So, we are going to change the restriction for the one proposed in [MA 13]

$$\sum_{i=1}^N \delta_i a_i^2.$$

Restrictions (a), (b) and (c) are linear, therefore the optimisation problem have similar difficulty as the one without these extra constraints. Restriction (d) turns the optimisation problem into integer programming which is computationally harder to resolve and some optimisation heuristic is needed. The optimisation problem with restriction (e) is also more difficult due to the non-linear constrain.

4 Analysis and simulations of conic portfolios

In this section we are going to implement the algorithms and methodologies that have been explained in previous chapters. We are going to carry out a comparison of models and perform an statistical analysis of them. Our computations are done with the programming language R, and are going to work with real data of Spanish market, in particular we will use the assets in the Ibex 35 index (see Appendix A).

In the following subsections, it will be shown how this theory is useful in practise. We are going to estimate the level of risk of the market and obtain optimal portfolios, answering the question of which assets to invest in.

4.1 Calibrating stress level

The aim of this part is to find the optimal stress level γ from historical data. We apply the algorithm explained in Section 2.2.2. The first step is generating the spread between bid and ask prices (see Figure 2). Once the bid and ask from market data have been computed, we create a theoretical bid and ask from formula (13). Finally, we get the γ that minimises the difference of prices along time in Figure 3.

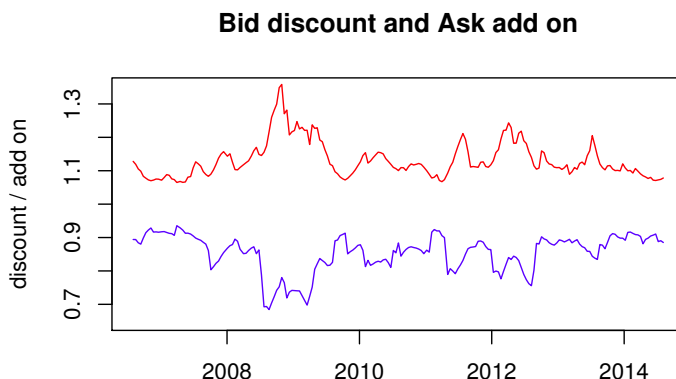


Figure 2: Bid and Ask spread in the IBEX 35 index (bid discount in blue and ask add on in red). We observe an increase in the spread in periods of recession or economic instability, specially between 2008 and 2010.

The γ values obtained along time are consistent with the financial situation of the Spanish economy between 2007 and 2014 (see Figure 3). We want to point out that there are some outliers between 2007 and 2008 due to housing bubble. The absence of liquidity in some sectors, specially affecting enterprises such as ACS and OHL, create an increase in stress levels. The relation between liquidity and stress level is well explained in [CGMS 10].

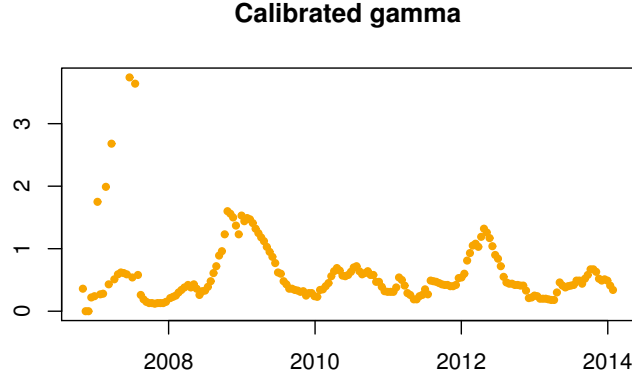


Figure 3: Calibrated stress levels from IBEX 35 index data. These values measure the risk in the Spanish market using historical data.

There are two main periods of stress in the data we are studying, one in 2008-2010 and one in 2012. The first period of stress appear due to the sever crisis that started in 2008. The second was caused by the rescue of some Spanish banks and the doubts generated about the solvency of Spain. Their consequence was an increase in the Spanish risk premium caught by our parameter γ . Hence, our simulations show empirically that the stress level is a risk parameter.

4.2 On the optimality of the calibration

Let us take periods of time between 01-01-2012 to 31-12-2014 and a 3 month horizon. We optimise our portfolio adapting the stress level in every optimisation date, calibrating it from market data. We compare the results with a non-adaptive optimisation, where the same γ is used at every optimisation date.

γ	Accumulated return
$\gamma_{\text{opt.}}$	0.663
0.20	0.599
0.40	0.587
0.60	0.551
0.80	0.639
1.00	0.539
1.20	0.581

With the optimal value of the stress level one gets a balanced portfolio where return and risk are compensated. However, the stress level can change during the investing period, affecting the optimality of the portfolio.

This optimisation method generates a portfolio that obtains greater profitability in comparison with other gamma levels. Looking at the table one sees that the accumulated return of the optimal portfolio is better than with all other gamma levels. However, the results are not significantly better, and there are some time periods where the calibrated gamma returns are not the best ones.

4.3 A comparison of conic and classical theory

In this section, we compute the optimal portfolio with calibrated market stress level. We compare our optimised conic portfolio with a benchmark of well known classical methods: a Markowitz portfolio (minimising the standard deviation of returns), a portfolio where risk is measured with the Expected Shortfall or TVaR and an equiponderated portfolio.

We want to see the profit of using each methodology in the accumulated portfolio return (see table below). The average return when applying each model and its quartiles as time evolves are also computed.

Portfolio model	Q1 quartile	Median	Q3 quartile	Mean ret.	Total ret.
Conic portfolio	−10.050	−5.485	0.579	−2.873	0.663
Markowitz	−10.800	−6.905	−2.437	−3.383	0.601
TVaR	−11.260	−6.620	−4.248	−6.502	0.434
Equiponderated	−9.346	−6.558	0.634	−3.998	0.583

The accumulated return is better for the conic portfolio model, where also the mean return is greater than in other methodologies. The same results can be appreciated with a boxplot of returns along time (see Figure 4).

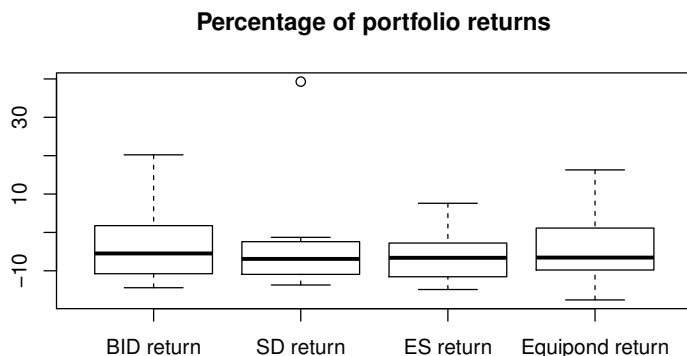


Figure 4: Graphical representation by quartiles of the percentage returns of a portfolio for different optimisation models.

We will also study the diversification of each of the portfolios. In [WP 93] Woerheide and Persson have conclude that the best indicator of portfolio diversification is the Herfindahl Concentration Index. This index is defined as

$$HI = \sum_{i=1}^N a_i^2, \quad (45)$$

where the a_i is the weight of the i -th asset in our portfolio. The following table show the average of portfolios concentration indices, where the lower the index the greater the diversification.

Portfolio model	Herfindahl Index
Conic portfolio	0.128
Markowitz	0.324
TVaR	0.417

4.4 On the distortion evaluation: MaxMinVar versus Wang Transform

We are going to compare the calculations that we have done on previous simulations, where a MaxMinVar distortion have been considered, with the usage of Wang Transforms. The Wang transform is a concave distortion, but not as severe as the MaxMinVar (see Figure 5).

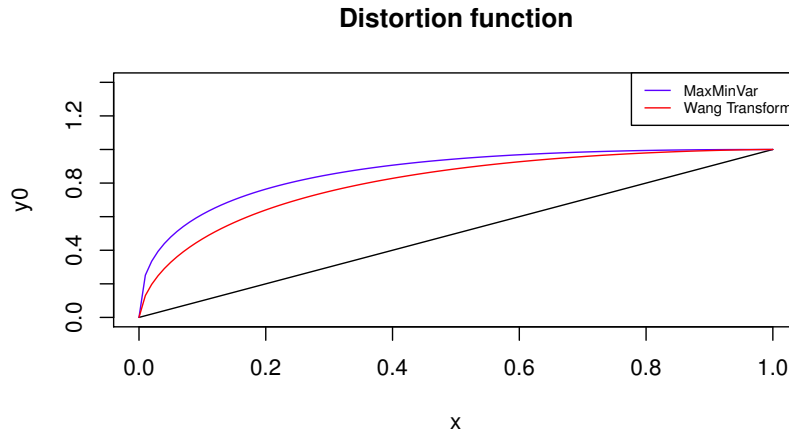


Figure 5: A distortion comparison between MaxMinVar and Wang transform. The x -axis represents the distribution probability before the distortion and the y -axis the final distribution after the distortion. We plot the non-distorted probability as an straight black line.

As the distortion is not the same, adapting our stress level to market risk means different γ values in the Wang transform and in the MaxMinVar distortion functions. In the following table there are the values of the calibrated stress level.

Period	MaxMinVar	Wang Transform
01-04-2012	0.37	0.65
01-07-2012	0.64	1.02
01-10-2012	0.86	1.31
31-12-2012	0.21	0.40
01-04-2013	0.11	0.22
01-07-2013	0.52	0.95
01-10-2013	0.31	0.60
31-12-2013	0.60	1.04
01-04-2014	0.24	0.51
01-07-2014	0.41	0.80
01-10-2014	0.31	0.67
31-12-2014	0.18	0.34

If we look at a particular date, for example 01-10-2012, we see in Figure 6 that both distortion functions generate almost the same probability distribution, each with its stress level ($\gamma = 0.86$ for the MaxMinVar and $\gamma = 1.31$ for the Wang transform).

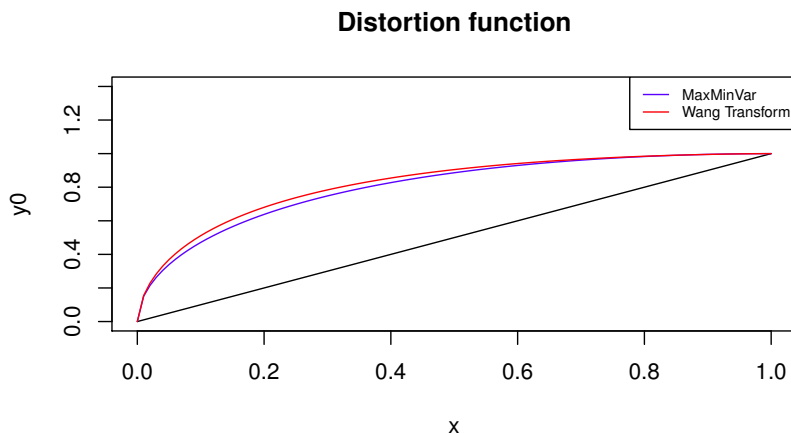


Figure 6: A distortion comparison between MaxMinVar with $\gamma = 0.86$ and Wang transform with $\gamma = 1.31$. The x -axis represents the distribution probability before the distortion and the y -axis the final distribution after the distortion. We plot the non-distorted probability as an straight black line.

We compute the profit of each portfolio at a 3 month horizon. The distortion taken is almost the same either if we apply the MaxMinVar or the Wang transform distortion,

as the stress level is calibrated to market data. However, the objective function is different and also are the returns. Looking at the returns, we can conclude that both distortions give rise to similar results. Figure 7 show a graphical representation of the quartiles of portfolio returns and Figure 8 their time evolution. The conclusion is that if the calibration is well done, the profit is not much affected for the usage of one distortion or the other.

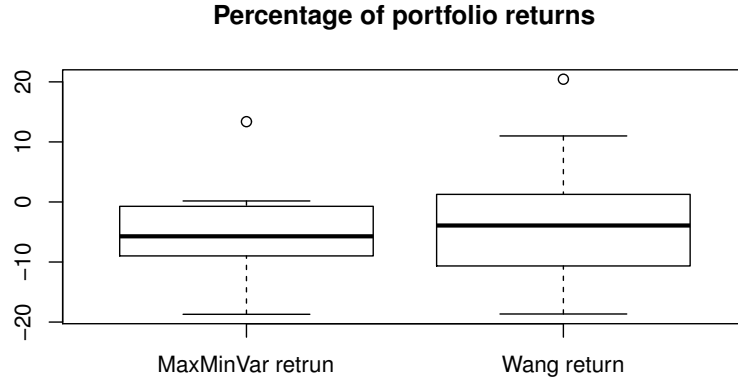


Figure 7: Graphical representation by quartiles of the portfolio returns with different distortions in the model. Calibrated γ from market risk is taken in both models.

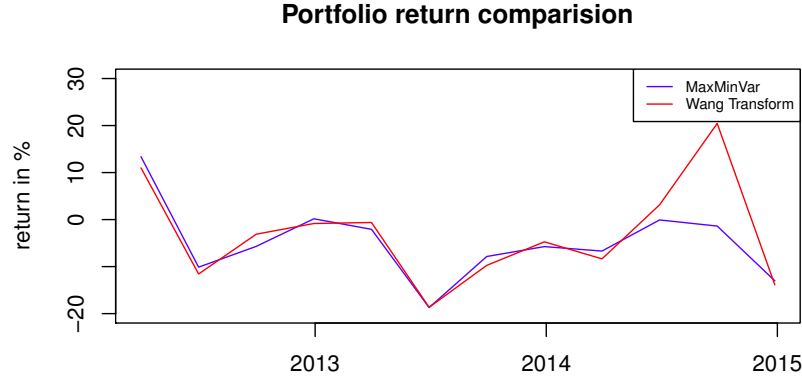


Figure 8: Time evolution of the portfolio returns with different distortions in the model. Calibrated γ from market data is taken in both models.

We also want to know which portfolio is more diversified, using the Herfindahl Concentration Index presented in formula (45). We obtain that both models, each with its

distortion function, are well diversified with a similar value of the Herfindahl Index:

Portfolio distortion	Herfindahl Index
MaxMinVar	0.122
Wang transform	0.120

We last compute the sensibility of both distortions to changes on the stress level (see Section 2.3) and we obtain different behaviours for each function. The values of the Wang transform sensibility are smaller than the ones of the MaxMinVar. Nevertheless, decreasing tendency of the sensibility as γ increases is greater in the MaxMinVar distortion (see Figures 9 and 10).

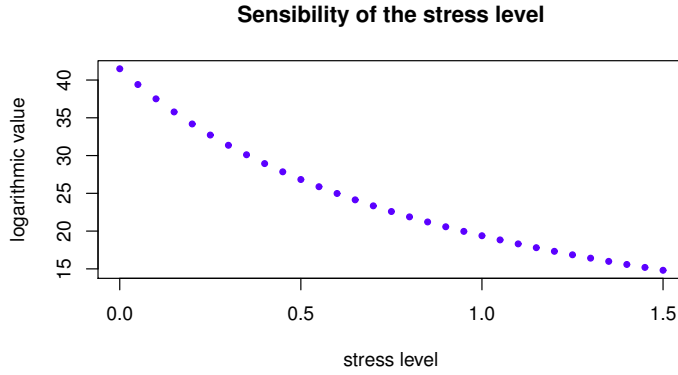


Figure 9: Sensibility of γ for the MaxMinVar distortion.

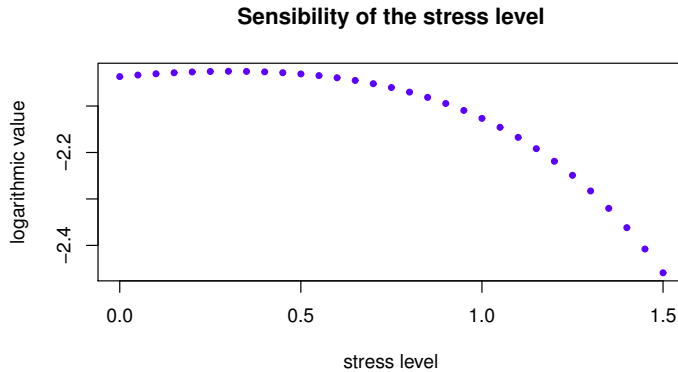


Figure 10: Sensibility of γ for the Wang transform distortion.

An interesting feature that we have found is that the sensibility decreases for large values of γ and thus this theory is particularly useful in periods of high stress on the markets.

4.5 Further calculations: restricted portfolio optimisation and Efficient Frontiers.

In this section we optimise the same problem as in previous sections, but with extra constraints. We bound the asset weights and we limit our portfolio to $K = 6$ number of assets. This last constraint complicates significantly the computations, as we are forced to solve an integer programming problem. To solve it we will use Differential Evolution, an stochastic method of optimisation [SP 97].

The heuristic of Differential Evolution is done in four steps, where the algorithm execute a range of random choices. Let $M, N, G \in \mathbb{R}$, where M is the population number, N is the number of assets (dimension of the objective function) and G is the generation number. We define a parameter vector $\vec{a}_{j,g} = (a_1, \dots, a_N)$ for $j = 1, \dots, M$ and $g = 1, \dots, G$.

1.- Initialisation:

- Define upper and lower bounds for each variable:

$$a_i^L \leq a_i \leq a_i^U, \quad i = 1, \dots, N.$$

- Select uniformly random values in the interval $[a_i^L, a_i^U]$ (for $g = 1$).

2.- Mutation:

- For a given parameter vector \vec{a}_j :
 - Randomly select vectors $\vec{a}_{r_1}, \vec{a}_{r_2}, \vec{a}_{r_3}$ with distinct subindices j, r_1, r_2, r_3 .
 - Add the weighted difference of two vectors to the third:

$$v_{j,g+1} = \vec{a}_{r_1} + \alpha \cdot (\vec{a}_{r_2} - \vec{a}_{r_3}).$$

with a constant $\alpha \in [0, 2]$.

3.- Recombination:

- Create the trial vector $\vec{u}_{j,g+1}$:

$$(u_{j,g+1})^{(i)} = \begin{cases} (v_{j,g+1})^{(i)} & \text{if } x_{i,j} \leq C, \\ (a_{j,g+1})^{(i)} & \text{if } x_{i,j} > C. \end{cases}$$

with $x_{ij} \sim U[0, 1]$ (uniform distribution). Hence, we choose $(v_{j,g+1})^{(i)}$ with probability $C \in [0, 1]$.

- $\vec{u}_{j,g+1}$ needs to have at least one coordinate different from $\vec{a}_{j,g}$.

4.- Selection:

- The vectors $\vec{a}_{j,g}$ and $\vec{u}_{j,g+1}$ are compared and we take the one with the lowest value of the objective function:

$$\vec{a}_{j,g+1} = \begin{cases} \vec{u}_{j,g+1} & \text{if } b(\vec{u}_{j,g+1}) \leq b(\vec{a}_{j,g}), \\ \vec{a}_{j,g} & \text{otherwise,} \end{cases}$$

where $b(\cdot)$ is the objective function (bid price).

We apply the Differential Evolution method to our stocks of the Ibex 35 index and obtain the following weights

Asset ticker	Weight
MAP.MC	0.100
ACS.MC	0.000
SAN.MC	0.266
REP.MC	0.000
TEF.MC	0.000
ITX.MC	0.000
BKT.MC	0.122
DIA.MC	0.240
ELE.MC	0.000
FCC.MC	0.000
FER.MC	0.278

The optimised bid price is 0.04694.

We can also implement the average return constrain, where we fix the value of the portfolio mean return. Then, for each value of the mean return (μ^*) that we assign to the problem, that is,

$$\sum_{i=1}^N a_i \mu_i = \mu^*,$$

where a_i is the weight of the i -th asset and μ_i its average return, we obtain an optimal portfolio. Combining all of them, we obtain the Efficient Frontier (see Figure 11).

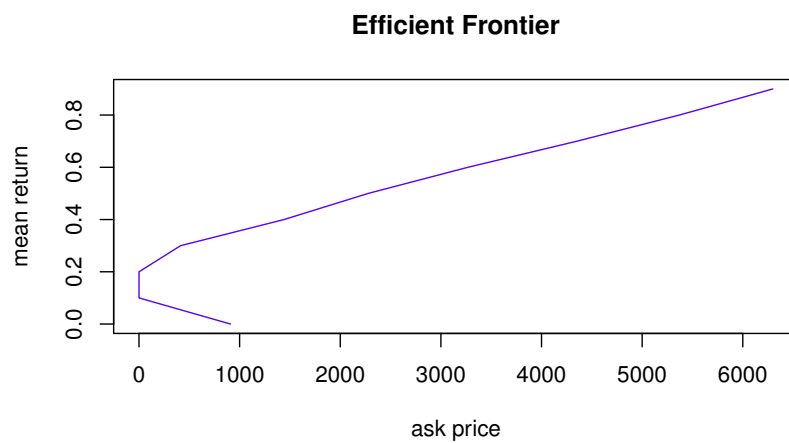


Figure 11: Efficient frontier with $\gamma = 0.1$.

5 Conclusion

We have done an statistical analysis of conic portfolio theory, an application of the bid price and coherent risk measures to portfolio optimisation. The main difficulty of this methodology is to deal with a tough nonlinear problem, where some optimisation heuristic is needed. The drawback of such probabilistic methods is that the optimal weights of the portfolio are not always the same, as random choices are part of the algorithm.

As the computational cost and the optimality approximation is quite stiff, a significant improvement in the results against other methods is necessary to compensate the effort, in execution time and mathematical tools. We have checked the return over a three months horizon compared with classical methods. In our simulations, it have been shown that the overall return of market value maximisation, a conic portfolio application, is better than Markowitz and ES optimisation. In addition, the portfolio obtained is shown to be well diversified, analysing diversification with Herfindahl Concentration Indices.

A key point when applying conic portfolio theory is selecting the distortion Ψ^γ used. Consciously of the importance of such function, we make a comparison analysis of the profits achieved when using a MaxMinVar distortion and a Wang transform distortion. Similar results are obtained in both cases if we take the stress level from market data. We have realised that greater levels of stress (γ) are needed in the Wang transform to have a similar distortion as in the MaxMinVar.

We have also given a method to select an optimal portfolio under size limitation and bounds on the weight constrains, using the heuristic optimisation method of Differential Evolution.

Summarising, this new theory brings an alternative optimisation model that behaves well with real data. Moreover, it provides a different approach to portfolio optimisation where a risk level from the investor is not needed, as the risk is calibrated from market data.

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Appendix A Ibex 35 Assets

Ticker	Asset name
ABE.MC	Abertis Infraestructuras S.A.
ACS.MC	Actividades de Construcción y Servicios, S.A.
ACX.MC	Acerinox, S.A.
AENA.MC	AENA
AMS.MC	Amadeus IT Holding SA
ANA.MC	Acciona, S.A.
BBVA.MC	Banco Bilbao Vizcaya Argentaria, S.A.
BKIA.MC	BANKIA SA
BKT.MC	Bankinter, S.A.
CABK.MC	CAIXABANK
DIA.MC	DIA
ELE.MC	Endesa, Sociedad Anonima
ENG.MC	Enagás, S.A.
FCC.MC	Fomento de Construcciones y Contratas, S.A.
FER.MC	Ferrovial, S.A.
GAM.MC	Gamesa Corporación Tecnológica S.A.
GAS.MC	Gas Natural SDG SA
GRF.MC	Grifols, S.A.
IAG.MC	International Airlines Group
IBE.MC	Iberdrola, S.A.
IDR.MC	Indra Sistemas, S.A.
ITX.MC	Industria de Diseno Textil Inditex SA
MAP.MC	Mapfre SA
MRL.MC	MERLIN PROP.
MTS.MC	ArcelorMittal SA
OHL.MC	Obrascon Huarte Lain SA
POP.MC	Banco Popular Espanol S.A.
REE.MC	Red Eléctrica Corporación S A.
REP.MC	Repsol, S.A.
SAB.MC	Banco de Sabadell, S.A.
SAN.MC	Banco Santander, S.A.
SCYR.MC	SACYR
TEF.MC	Telefónica, S.A.
TL5.MC	Mediaset España Comunicación, SA
TRE.MC	Tecnicas Reunidas, S.A.

Appendix B Glossary

A:

Adjusted closing price: end of day price modified to distribute discontinuities, such as splits or dividends, among time.

Arbitrage: operating in an economy or different markets in order to take advantage of some incoherence in prices that permits gaining money with zero risk. It is sometimes called “free lunch”.

Ask price: price at which the market is willing to sell an asset or, equivalently, at which a person can buy an asset from the market.

Asset: an item of property, regarded as having value, owned by a person or company. In a financial context, an asset can be a product such as a stock, a derivative, a bond...

B:

Bid price: price at which the market is willing to buy an asset or, equivalently, at which a person can sell an asset to the market.

Bond: a debt obligation issued by a government, a public company or a private one that will be paid at future dates with a fixed rate of interests.

C:

Call: an option that gives the right to purchase an underlying asset at a future date and at a fixed price (strike).

Cash flow: The total amount of money being transferred from a transaction, for example the purchase and sell of an stock.

D:

Derivative: a financial product (such as a future, an option or a swap) whose value derives from the value of an underlying asset.

Diversification: the process of enlarging or varying the range of products in a portfolio.

F:

Forward/Future: an arrangement for an exchange of products at a future date and a fixed price. It is a type of derivative.

K:

Kurtosis: a measure of the extremal data on a distribution. It takes into account the tail shape, width of peaks and the lack of shoulders.

L:

Liquidity: degree to which an asset can be quickly bought or sold in the market without affecting the asset's price. It is very much related with the volume of negotiation.

O:

Option: a contract that gives the right to sell (put option) or to purchase (call option) an underlying asset at a future date with price (strike) settled at the beginning of the contract. There are other more complicated options usually referred as exotic. It is a type of derivative.

S:

Share: one of the equal parts into which a company's capital is divided. It is a certificate of ownership, entitling the holder to a proportion of the profits.

Skewness: a measure of the asymmetry of a probability distribution with respect to its mean value.

Spread: difference between the bid and ask prices. It is very much related with the liquidity of the market and the moment of the economic cycle. It can also mean the factor added to a price to be covered over some possibility of losses.

Stock: it is used as a synonym of share.

Strike: fixed price at which an option can be exercised. A price fixed now at which you can sell or buy an asset in the future.

Swap: an exchange of assets between two parties at a future date. For example, an Interest Rate Swap (IRS) exchange a fixed interest rate for a floating one.

P:

Portfolio: A range of investments held by a person or organisation.

Put: an option that gives the right to sell an underlying asset at a future date and at a fixed price (strike).

V:

Volatility: a statistical measure of the dispersion of returns for a given asset.

Appendix C R programs description

In the R code, we define the stocks and get the assets real price within the period selected, in our case from 01-01-2012 to 31-12-2014. We define the objective and the distortion functions and, then, we are ready to start the optimisation process (in line 104).

To proceed with the optimisation, we transform the downloaded prices into returns. Then, we calibrate γ from market data with least squares optimisation. Once we have the value of γ , we optimise the portfolio with the market stress level.

Finally, we calculate the accumulated returns and show the graphical representation of the solutions.

```
1 #####
2 ##LONG ONLY OPTIMISATION PROBLEM
3 ##Using PortfolioAnalytics package
4 ##Algorithm to find the best portfolio with a setted stress level
5 ##Autor: Sergi Ferrer and Argimiro Arratia
6 #####
7
8 #####LOAD PACKAGES#####
9
10 library(PortfolioAnalytics)
11 library(DEoptim)
12 library(GenSA)
13 library(ROI)
14 library(ROI.plugin.quadprog)
15
16 library(plyr)
17 library(quantmod)
18
19 library(ineq)
20
21 #####END LOADING PACKAGES#####
22
23 #####DEFINE DATA: STOCK NAMES AND DATES#####
24
25 ## Data of market "IBEX" (index) from yahoo.finance
26 stocks11=c("MAP.MC", "ACS.MC", "SAN.MC", "REP.MC", "TEF.MC", "ITX.MC",
27           "BKT.MC", "GAM.MC", "ELE.MC", "FCC.MC", "FER.MC")
28
29 ##All IBEX:
30 stocks33 <- c("ABE.MC", "ACS.MC", "ACX.MC", "AMS.MC", "ANA.MC",
```

```
31         "BBVA.MC", "BKIA.MC", "BKT.MC", "CABK.MC", "DIA.MC",
32         "ELE.MC", "ENG.MC", "FCC.MC", "FER.MC", "GAM.MC",
33         "GAS.MC", "GRF.MC", "IAG.MC", "IBE.MC", "IDR.MC",
34         "ITX.MC", "MAP.MC", "MTS.MC", "OHL.MC", "POP.MC",
35         "REE.MC", "REP.MC", "SAB.MC", "SAN.MC", "SCYR.MC",
36         "TEF.MC", "TL5.MC", "TRE.MC")
37
38 stocksMadan<- c("GE", "INTC", "IBM", "JNJ", "APPL")
39
40
41 dateI="2012-01-01"; dateF="2014-12-31"
42 a_dateI <- seq(as.Date(dateI, '%Y-%m-%d'),
43               as.Date(dateF, '%Y-%m-%d')-91, by = 91)
44 a_dateF <- seq(as.Date(dateI, '%Y-%m-%d')+91,
45               as.Date(dateF, '%Y-%m-%d'), by = 91)
46 dates <- list()
47 for (i in 1:length(a_dateI)) {
48   dates[[i]] <- c(a_dateI[i], a_dateF[i])
49 }
50
51 #####END DATA#####
52
53 #####DEFINITION OF FUNCTION AND DISTORTION#####
54
55 #Definition of the concave distortion to be used
56 Psi_MaxMinVar <- function(gamma,x) 1-(1-x^(1/(1+gamma)))^(1+gamma)
57 Psi_Wang <- function(gamma,x) pnorm(qnorm(x) + gamma)
58 Psi <- Psi_MaxMinVar
59
60 #Objective function. Parameters: R for returns and
61 # weights for asset proportions.
62 Obj<-function(R,weights,gamma=0.38) {
63   rw<-R%*%weights
64   ret<-sort(rw)
65   vl<-nrow(R)
66   m<-1:vl/vl
67   c<-Psi(gamma,m)-Psi(gamma,m-1/vl)
68   c[is.nan(c)]<-0
69   return(-sum(c*ret))
70 }
71
72 #Theoretical bid price depending on the stress level
```

```
73 BID <- function(gamma,Ret) { #Ret in increasing order
74   #Obtention of the coefficients depending on gamma
75   vl<-length(Ret)
76   m<-1:vl/vl
77   c<-Psi(gamma,m)-Psi(gamma,m-1/vl)
78   c[is.nan(c)]<-0
79   #Portfolio bid return
80   BID<-sum(Ret*c)
81 }
82
83 #Theoretical ask price depending on the stress level
84 ASK <- function(gamma,Ret) { #Ret in increasing order
85   #Obtention of the coefficients depending on gamma
86   vl<-length(Ret)
87   m<-1+1/vl-1:vl/vl
88   c<-Psi(gamma,m)-Psi(gamma,m-1/vl)
89   c[is.nan(c)]<-0
90   #Portfolio ask return
91   ASK<-sum(Ret*c)
92 }
93
94 #Parameters
95 stocks=stocks11
96 n<-length(stocks)
97 w_equip<-seq(1/n,1/n,length.out = n)
98
99 #####END DEFINITION OF FUNCTIONS#####
100
101 #####PORTFOLIO OPTIMISATION FUNCTION#####
102
103 ##OPTIMISATION FUNCTION:
104 optimization_comparision<- function(date) {
105   #####Get data#####
106   # Parameters:
107   dateIni<-date[1] # Initial date
108   dateFin<-date[2] # Final date
109   N_assets <- length(stocks) # Number of assets
110   T_return <- 90 # Horizont of the return
111   #Get data from yahoo finance
112   data.env<-new.env()
113   l_ply(stocks, function(sym) try(getSymbols(sym,src="yahoo",
114     from=dateIni,to=dateFin,env=data.env),silent=T))
```

```
115 stocks <- stocks[stocks %in% ls(data.env)]
116 ### loop through get Ad.Close, compute daily log-return
117 ### and merge all stocks
118 returns <- xts()
119 for(i in seq_along(stocks)) {
120   sym <- stocks[i]
121   returns <- merge(returns,
122                     periodReturn(Ad(get(sym,envir=data.env)),
123                                   period="daily",type = "log"))
124 }
125 returns <- returns[-1] # remove first row == 0
126 returns[is.na(returns)]<-0 # put zeros in place of NA
127 colnames(returns) <- paste(stocks, ".logret", sep="")
128 #####End get data#####
129
130 #####Calibration of gamma#####
131 data.env_cal<-new.env()
132 l_ply(stocks, function(sym) try(getSymbols(sym,src="yahoo",
133   from=dateIni-300,to=dateFin,env=data.env_cal),silent=T))
134
135 #Price of every asset definition
136 X<-list()
137 for(i in 1:N_assets) {
138   X[[i]]<-data.env_cal[[stocks[i]]][,6]
139 }
140
141 #Volume of every asset definition
142 V<-list()
143 for(i in 1:N_assets) {
144   V[[i]]<-data.env_cal[[stocks[i]]][,5]
145 }
146
147 #Parameters and initialization of variables
148 cont<-seq(1,length(X[[1]])-92,by=10)
149 b<-matrix(0,N_assets,length(cont))
150 a<-matrix(0,N_assets,length(cont))
151 b_disc<-matrix(0,N_assets,length(cont))
152 a_add<-matrix(0,N_assets,length(cont))
153 m<-matrix(0,N_assets,length(cont))
154 vol<-matrix(0,N_assets,length(cont))
155 reward<-matrix(0,N_assets,length(cont))
156 N_vol<-dim(returns)[1]-3
```

```
157
158 #Generate the bid, ask and average prices for every period
159 for(i in 1:N_assets) { # i = Asset index
160   for(j in cont) { # j = Period of time index
161     #Volume
162     vol[i,1+j/10]<-mean(V[[i]][j:(j+90)]) # volume
163     #Prices
164     b[i,1+j/10]<-min(X[[i]][j:(j+90)]) # bid price
165     a[i,1+j/10]<-max(X[[i]][j:(j+90)]) # ask price
166     m[i,1+j/10]<-mean(X[[i]][j:(j+90)]) # average price
167     #Discounts
168     b_disc[i,1+j/10]<-b[i,1+j/10]/m[i,1+j/10] # bid discount
169     a_add[i,1+j/10]<-a[i,1+j/10]/m[i,1+j/10] # ask add on
170     #Return
171     reward[i,1+j/10] <-
172       ((as.numeric(X[[i]][j+90])/as.numeric(X[[i]][j]))-1)
173   }
174 }
175
176 #Reward of the assets at an horizon of T_return days
177 cont<-1:(length(X[[1]])-T_return-2)
178 asset_reward<-matrix(0,N_assets,length(cont))
179 asset_vol<-matrix(0,N_assets,length(cont))
180 asset_w<-matrix(0,N_assets,length(cont))
181 for(i in 1:N_assets) {
182   for(j in cont) {
183     #Return
184     asset_reward[i,j] <-
185       ((as.numeric(X[[i]][j+T_return])/as.numeric(X[[i]][j]))-1)
186     #Volume
187     asset_vol[i,j]<-mean(V[[i]][j:(j+T_return)])
188   }
189 }
190
191 #Computation of the weights for every asset
192 #using the volume of negotiation
193 for(i in 1:N_assets) {
194   for(j in cont) {
195     asset_w[i,j]<-asset_vol[i,j]/sum(asset_vol[,j])
196   }
197 }
198
```

```
199   #Portfolio return: asset returns times the weights
200   ReturnF<-c()
201   for(j in cont) {
202     ReturnF[j]<-sum(asset_w[,j]*asset_reward[,j])
203   }
204
205   #Parameters
206   n<-ncol(b_disc)
207   w<-matrix(0,N_assets,n)
208   bid<-c()
209   ask<-c()
210
211   #Computation of the daily weights of every asset
212   for(i in 1:N_assets) {
213     for(j in 1:n) {
214       w[i,j]<-vol[i,j]/sum(vol[,j])
215     }
216   }
217
218   #Combination of the assets in the IBEX35 index
219   #to get the market bid and ask
220   for(j in 1:n) {
221     bid[j]<-sum(w[,j]*b_disc[,j])
222     ask[j]<-sum(w[,j]*a_add[,j])
223   }
224
225   #Optimization of the stress level with the returns obtained:
226
227   #Gamma parameters:
228   Gamma_ini <- 0
229   Gamma_fin <- 4
230   cont_gamma <- seq(Gamma_ini,Gamma_fin,by=0.01)
231   optimal_gamma<-0
232   min<-1/0
233
234   Return_sort<-sort(ReturnF)
235
236   #Optimization loop:
237   for (gamma in cont_gamma) { # Loop on the gamma to be optimized
238     #Theoretical bid and ask computation:
239     bid_theoretical <- BID(gamma,Return_sort)
240     ask_theoretical <- ASK(gamma,Return_sort)
```



```
241   #Least squares optimization on bid and ask:
242   L_squares <- (bid_theoretical-last(bid))^2 +
243               (ask_theoretical-last(ask))^2
244   if(L_squares<min) {
245     min <- L_squares
246     optimal_gamma <- gamma
247   }
248 }
249 #Result
250 print(paste("The value for the stress level is: ",optimal_gamma))
251 #####End Calibration of gamma#####
252
253 #####Optimisation#####
254
255 # initialize portfolio spec + constraints (constraint object)
256 portf <- portfolio.spec(assets = stocks)
257 portf <- add.constraint(portfolio = portf, type = "long_only")
258 portf <- add.constraint(portf, type="weight_sum",
259                        min_sum=0.99, max_sum=1.01)
260
261 ##Add objective to the portfolio object
262 portf.bid <- add.objective(portfolio=portf, # our portfolio object
263                          type="risk", # the kind of objective
264                          name="Obj", # the function to optimise
265                          enabled=TRUE,
266                          arguments=list(gamma=optimal_gamma)
267                          )
268
269 ##Other objective for testing
270 minSD.portfolio <- add.objective(portfolio=portf,
271                                type="risk",
272                                name="StdDev")
273
274 minES.portfolio <- add.objective(portfolio=portf,
275                                type="risk",
276                                name="ES")
277
278 # Return data
279 p_returns <- c()
280 data.env_adj<-new.env()
281 l_ply(stocks, function(sym) try(getSymbols(sym,src="yahoo",
282                                           from=as.Date(dateFin, '%Y-%m-%d'),
```

```
283         to=as.Date(dateFin,'%Y-%m-%d')+91,env=data.env_adj)
284         ,silent=T))
285     #funcio de calcul del ad inicial i final
286     get_adjusted <- function(x) {
287         adjust <- Ad(get(x,envir=data.env_adj))
288         out <- c(adjust[1],last(adjust))
289         return(out)
290     }
291     asset_price <- aapply(cbind(stocks),1,get_adjusted)
292
293     # Run the optimization to generate sample portfolio
294     #optimize_method: "GenSA"(generalized simulated annealing)
295     N_heuristic<-100 #Number of simulations of the heuristic
296     p_returns[1]<-0
297     for (i in 1:N_heuristic) {
298         maxBid.opt <- optimize.portfolio(R = returns, portfolio=portf.bid,
299                                         optimize_method = "GenSA",
300                                         search_size = 1500,
301                                         trace = TRUE)
302         p_returns[1] <- p_returns[1] + (1/N_heuristic)*
303             (sum(maxBid.opt$weights*asset_price[,1])
304              /sum(maxBid.opt$weights*asset_price[,2])-1)
305     }
306
307
308     maxSD.opt <- optimize.portfolio(R = returns, portfolio=minSD.portfolio,
309                                     optimize_method = "ROI",
310                                     search_size = 1500,
311                                     trace = TRUE)
312     maxES.opt <- optimize.portfolio(R = returns, portfolio=minES.portfolio,
313                                     optimize_method = "ROI",
314                                     search_size = 1500,
315                                     trace = TRUE)
316
317     #####End Optimisation#####
318
319     #####Return and concentration index#####
320
321     #Return calculation
322     p_returns[2] <- sum(maxSD.opt$weights*asset_price[,1])/
323                     sum(maxSD.opt$weights*asset_price[,2])-1
324     p_returns[3] <- sum(maxES.opt$weights*asset_price[,1])/
```

```

325         sum(maxES.opt$weights*asset_price[,2])-1
326 p_returns[4] <- sum(w_equip*asset_price[,1])/
327         sum(w_equip*asset_price[,2])-1
328
329 #Herfindahl index
330 diversification<-list(BID=0,SD=0,ES=0)
331 diversification$BID<-Herfindahl(maxBid.opt$weights,
332         parameter = 1, na.rm = TRUE)
333 diversification$SD<-Herfindahl(maxSD.opt$weights,
334         parameter = 1, na.rm = TRUE)
335 diversification$ES<-Herfindahl(maxES.opt$weights,
336         parameter = 1, na.rm = TRUE)
337
338 #####End return and concentration index#####
339
340 # Output
341 return(list(BID=maxBid.opt,SD=maxSD.opt,ES=maxES.opt,
342         RETURNS=p_returns,DIVERS=diversification))
343 }
344
345 #####END PORTFOLIO OPTIMISATION FUNCTION#####
346
347 #####COMPUTATION OF THE PORTFOLIO#####
348
349 # Obtain the optimal portfolio
350 opt.portfolios<-llply(dates,optimization_comparision)
351
352 # Obtain the cumulative return
353 cum_ret <- list(BID=1,SD=1,ES=1,EQPON=1)
354 ret_value <- list(BID=c(),SD=c(),ES=c(),EQPON=c())
355 div_value <- list(BID=c(),SD=c(),ES=c(),EQPON=c())
356 for (i in 1:length(opt.portfolios)) {
357     cum_ret$BID <- cum_ret$BID*(1+opt.portfolios[[i]]$RETURNS[1])
358     cum_ret$SD <- cum_ret$SD*(1+opt.portfolios[[i]]$RETURNS[2])
359     cum_ret$ES <- cum_ret$ES*(1+opt.portfolios[[i]]$RETURNS[3])
360     cum_ret$EQPON <- cum_ret$EQPON*(1+opt.portfolios[[i]]$RETURNS[4])
361     ret_value$BID[i] <- opt.portfolios[[i]]$RETURNS[1]*100
362     ret_value$SD[i] <- opt.portfolios[[i]]$RETURNS[2]*100
363     ret_value$ES[i] <- opt.portfolios[[i]]$RETURNS[3]*100
364     ret_value$EQPON[i] <- opt.portfolios[[i]]$RETURNS[4]*100
365     div_value$BID[i] <- opt.portfolios[[i]]$DIVERS[[1]]
366     div_value$SD[i] <- opt.portfolios[[i]]$DIVERS[[2]]

```

```

367   div_value$ES[i] <- opt.portfolios[[i]]$DIVERS[[3]]
368 }
369
370 divers<-list(BID=mean(div_value$BID),
371             SD=mean(div_value$SD),ES=mean(div_value$ES))
372
373 #####END COMPUTATION OF THE PORTFOLIO#####
374
375
376 #####PLOT OF THE RETURNS#####
377
378 plot(a_dateF,ret_value$BID,type="l", col = "blue",
379      main="Portfolio return comparision", ylab = "return in %",
380      xlab = "", ylim = c(-20,30))
381
382 legend('topright', c("Bid optimization","SD optimization",
383                      "ES optimization","Equiponderated portfolio") ,
384        lty=1, col=c('blue', 'red', 'black',' orange'), cex=.65)
385
386 lines(a_dateF,ret_value$SD, col = "red")
387 lines(a_dateF,ret_value$ES, col = "black")
388 lines(a_dateF,ret_value$EQPON, col = "orange")
389
390 boxplot(ret_value$BID,ret_value$SD,ret_value$ES, ret_value$EQPON,
391         names = c("BID return","SD return","ES return","Equipond return"),
392         main = "Percentage of portfolio returns")
393
394 #####END PLOT OF THE RETURNS#####
395

```